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Periodic Quadratic Spline Interpolant of Minimal Norm

GÜNTER MEINARDUS

Fachbereich Mathematik, University of Siegen, D-5900 Siegen 21, West Germany

AND

G. D. TAYLOR*

Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523

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1. INTRODUCTION

Let $K = \{x_0, ..., x_n\}$ be a partition of $[0, 1]$, $0 = x_0 < x_1 < \cdots < x_n = 1$. The class of all periodic quadratic splines with respect to this partition, $S_3(K)$, is defined to be the set of all $s \in C^1[0, 1]$ such that $s(x)$ restricted to $[x_{\nu-1}, x_\nu]$ is a real algebraic polynomial of degree 2 and $s^{(j)}(x_0) = s^{(j)}(x_n)$, $j = 0, 1$. It is well known that each $s \in S_3(K)$ is uniquely determined by its values at $x_1, ..., x_n$ if and only if n is odd [2]. Thus, we shall assume that n is odd henceforth. Define $h_\nu = x_\nu - x_{\nu-1}$; $h_{\nu+n} = h_\nu$, $\nu = 1, ..., n$, $H = \operatorname{diag}(h_1, ..., h_n)$ a $n \times n$ matrix and $\Gamma(H)$ a mapping from $C[0, 1]$ with the uniform norm, $\| h \| = \max\{| h(t)|: 0 \leqslant t \leqslant 1\}$, to $S_3(K)$ by

$$\Gamma(H) f = s_f(x),$$

where $s_f(x)$ is the unique spline in $S_3(K)$ satisfying $s_f(x_i) = f(x_i)$, $i = 1, ..., n$. $\Gamma(H)$ is a projection operator onto $S_3(K)$. Norming $S_3(K)$ with the uniform norm, then we may define

$$\| \Gamma(H)\| = \sup_{\|f\| \leqslant 1} \| \Gamma(H) f \|.$$

The result that we wish to prove is:

THEOREM 1. *Let $n \geqslant 3$ be an odd integer. Then*

$$\inf_{K \in \mathscr{K}} \| \Gamma(H)\| = \| \Gamma(\hat{H})\| = (n + 1)/2,$$

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where $\hat{H} = \mathrm{diag}(1/n, 1/n, ..., 1/n)$ *is the matrix corresponding to the partition* $\hat{K}$ *of equally spaced knots,* $\mathscr{K}$ *denotes the set of all partitions of* $[0, 1]$ *into* n *distinct subintervals. In addition,* $\| \Gamma(H) \|$ *is the only global minimum for this problem.*

That is, the periodic quadratic spline interpolant (projector operator) of minimal norm corresponds uniquely to the case of equally spaced knots for n odd. For a survey of results concerning projections of minimal norm see [1].

2. Notation and Proof

The method of establishing this result will be to rephrase this problem in a vector-valued polynomial setting as done recently in [3]. The advantage of this approach is that it allows us to treat this problem in a purely algebraic manner. Specifically, let Π_2 denote the collection of all real-valued polynomials of degree 2 or less and let $\mathbf{\Pi}_2{}^n$ denote the class of all n-dimensional vector-valued polynomials of degree $\leqslant 2$, i.e., $\mathbf{q} \in \mathbf{\Pi}_2{}^n$ if and only if $\mathbf{q}(t) = (q_1(t), ..., q_n(t))^T$ with $q_i(t) \in \Pi_2$ for $i = 1, ..., n$. Norm $\mathbf{\Pi}_2{}^n$ with the norm $\| \mathbf{q} \|_n = \max_{1 \leqslant i \leqslant n} \| q_i \| = \max_{1 \leqslant i \leqslant n}(\max_{0 \leqslant t \leqslant 1} | q_i(t) |)$. Define the $n \times n$ matrices A and T by $A = \mathrm{diag}(\alpha_1, ..., \alpha_n)$ where $\alpha_\nu = h_{\nu+1}/h_\nu$ for all ν and

$$T = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & & & \cdot \\ & & \cdot & & \\ & & & \cdot & \\ & & & 1 & 0 \end{pmatrix}.$$

Note that $T^*T = I$, $T^n = I$, $A = T^*HTH^{-1}$ where the asterisk denotes the Hermitian conjugate.

For each $s \in S_3(K)$ set $s(x) = s_\nu(x)$ for $x \in [x_{\nu-1}, x_\nu]$, $\nu = 1, ..., n$. Define a mapping $\bar{T}$ of $S_3(K)$ into $\mathbf{\Pi}_2{}^n$ by $\bar{T}s = \mathbf{q} = (q_1, ..., q_n)^T$ where $q_\nu(t) = s_\nu(x_{\nu-1} + t(x_\nu - x_{\nu-1}))$, $\nu = 1, ..., n$, $t \in [0, 1]$. Let $\bar{S}_3(K)$ denote the image of $S_3(K)$ under $\bar{T}$. It is easily seen that $\mathbf{q} \in \bar{S}_3(K)$ if and only if $\mathbf{q}^{(j)}(0) = TA^j\mathbf{q}^{(j)}(1)$, $j = 0, 1$ where $\mathbf{q}_1^{(j)}(t) = (q_1^{(j)}(t), ..., q_n^{(j)}(t))^T$ and $\bar{T}$ is an isomorphic isometry between $S_3(K)$ and $\bar{S}_3(K)$. Let $\mathbf{q}_1, ..., \mathbf{q}_n$ be a set of n functions in $\mathbf{\Pi}_2{}^n$ where $\mathbf{q}_i = (q_{1i}, ..., q_{ni})^T$, $i = 1, ..., n$. Define the $n \times n$ matrix $W(t, A)$ with respect to this set by

$$\begin{aligned} W(t, A) &= (\mathbf{q}_1(t), ..., \mathbf{q}_n(t)) \\ &= \begin{pmatrix} q_{11}(t) & \cdots & q_{1n}(t) \\ \vdots & & \vdots \\ q_{n1}(t) & \cdots & q_{nn}(t) \end{pmatrix}. \end{aligned}$$

If $\{\mathbf{q}_1, ..., \mathbf{q}_n\}$ is a basis for $\bar{S}_3(K)$ then we shall call $W(t, A)$ a basis matrix.

Note that $\{\mathbf{q}_1, ..., \mathbf{q}_n\}$ is a basis for $\bar{S}_3(K)$ if and only if $\{\bar{T}^{-1}\mathbf{q}_1, ..., \bar{T}^{-1}\mathbf{q}_n\}$ is a basis for $S_3(K)$. If $W(t, A)$ is a basis matrix then for $\mathbf{c} = (c_1, ..., c_n)^T \in \mathbb{R}^n$, $\mathbf{q}(t) = W(t, A)\,\mathbf{c} = \sum_{j=1}^{n} c_j \mathbf{q}_j(t) \in S_3(K)$. The following facts are easily proven for the matrix $W(t, A)$:

LEMMA 1. *Let B be a nonsingular $n \times n$ matrix and set $W(t, A) \cdot B = (\mathbf{v}_1(t), ..., \mathbf{v}_n(t))$ where $W(t, A)$ is a basis matrix. Then $\{\mathbf{v}_1(t), ..., \mathbf{v}_n(t)\}$ is a basis for $\bar{S}_3(K)$.*

LEMMA 2. *If $W^{(j)}(0, A) = TA^j W^{(j)}(1, A)$, $j = 0, 1$ and there exists $\tau \in [0, 1]$ such that $W^{-1}(\tau, A)$ exists then $W(t, A)$ is a basis matrix.*

LEMMA 3. *Fix $\tau \in [0, 1]$. Then corresponding to each $\mathbf{y} \in \mathbb{R}^n$ there exists a unique $\mathbf{q} \in \bar{S}_3(K)$ such that $\mathbf{q}(\tau) = \mathbf{y}$ if and only if $W^{-1}(\tau, A)$ exists for each basis matrix $W(t, A)$.*

LEMMA 4. *Let n be odd integer. Then $W^{-1}(1, A)$ exists for each basis matrix $W(t, A)$.*

Define the mapping $\bar{\Gamma}(H)$ of $C[0, 1]$ into $\bar{S}_3(K)$ by

$$\bar{\Gamma}(H) f = W(t, A)\,\mathbf{f},$$

where $\mathbf{f} = (f(x_1), ..., f(x_n))^T$ and $W(t, A) = (\mathbf{q}_1(t), ..., \mathbf{q}_n(t))$ is a basis matrix that satisfies: (1) $W(1, A) = I$, (2) $W(0, A) = T$ and (3) $W'(0, A) = TAW'(1, A)$. That such a basis matrix exists follows from Lemmas 1, 2, and 4. Observing that $\bar{\Gamma}(H) f = \bar{T}\Gamma(H) f$ for all $f \in C[0, 1]$, we have that

$$\begin{aligned} \| \Gamma(H) \| = \| \bar{\Gamma}(H) \| &= \sup_{\|f\| \leqslant 1} \| W(t, A)\,\mathbf{f} \|_n \\ &= \sup_{\|f\| \leqslant 1} \left(\max_{1 \leqslant i \leqslant n} \| f(x_1)\, q_{i1}(t) + \cdots + f(x_n)\, q_{in}(t) \| \right) \\ &= \max_{0 \leqslant t \leqslant 1} \left| \max_{1 \leqslant i \leqslant n} (|q_{i1}(t)| + \cdots + |q_{in}(t)|) \right| \\ &= \max_{0 \leqslant t \leqslant 1} \| W(t, A) \|_\infty , \end{aligned}$$

where $\| B \|_\infty$ denotes the maximum absolute row sum of B.

Thus, we wish to estimate

$$\inf_{A \in \mathfrak{A}} \left(\max_{0 \leqslant t \leqslant 1} \| W(t, A) \|_\infty \right),$$

where $\mathfrak{A} = \{A\colon A = \operatorname{diag}(\alpha_1, ..., \alpha_n)$ with $\alpha_i > 0$ for all i and $\prod_{i=1}^{n} \alpha_i = 1\}$. Since the set $\mathfrak{A}$ is in a 1–1 correspondence with $\mathscr{K}$ this is a rephrasing of the minimal periodic quadratic spline projection problem. Thus, Theorem 1 follows from

THEOREM 2. *Let n be an odd integer. Then*

$$\max_{0\leq t\leq 1} \| W(t, A)\|_\infty = \frac{1}{2} + \frac{1}{2}\left(\max_{1\leq j\leq n} h_j\right) \sum_{i=1}^{n} \frac{1}{h_i} \geq \max_{0\leq t\leq 1} \| W(t, I)\|_\infty = \frac{n+1}{2},$$

where $A = I$ is the only global minimum, ($A = I$ is equivalent to equally spaced knots.)

Proof. Suppose $W(t, A) = (\mathbf{q}_1, ..., \mathbf{q}_n)$ where $\mathbf{q}_i = (q_{1i}, ..., q_{ni})^T$, $i = 1, ..., n$. Since t^2, $(1-t)^2$, $2t(1-t)$ forms a basis for Π_2 we may write each $q_{ij}(t) = b_{ij}^{(1)} t^2 + b_{ij}^{(2)}(1-t)^2 + 2b_{ij}^{(3)} t(1-t)$ and hence

$$W(t, A) = t^2 M + (1-t)^2 L + 2t(1-t) K,$$

where M, L and K are $n \times n$ real matrices. Since $W(1, A) = I$ and $W(0, A) = T$ we must have

$$W(t, A) = t^2 I + (1-t)^2 T + 2t(1-t) K.$$

Differentiating with respect to t, gives

$$W'(t, A) = 2tI - 2(1-t) T + 2(1-2t) K$$

so that $W'(0, A) = -2T + 2K$ and $W'(1, A) = 2I - 2K$. Thus, K must satisfy

$$-T + K = TA(I - K).$$

Solving for K, using the identities $TA = HTH^{-1}$ and $T^*T = I$ gives

$$K = HT(I + T)^{-1}(H^{-1} + T^* H^{-1} T).$$

Next, observe that since $T^n = I$,

$$(I + T)(I - T + T^2 - \cdots + (-1)^{n-1} T^{n-1}) = I + (-1)^{n-1} I = 2I$$

since n is odd. Thus,

$$2T(I + T)^{-1} = I + \sum_{\nu=1}^{n-1} (-1)^{\nu-1} T^\nu$$

and

$$2K = H\left(I + \sum_{\nu=1}^{n-1} (-1)^{\nu-1}T^{\nu}\right)(H^{-1} + T^*H^{-1}T).$$

Noting that

$$H = \operatorname{diag}(h_1, \dots, h_n) \qquad \text{and}$$

$$(H^{-1} + T^*H^{-1}T) = \operatorname{diag}\left(\frac{1}{h_1} + \frac{1}{h_2}, \dots, \frac{1}{h_{n-1}} + \frac{1}{h_n}, \frac{1}{h_n} + \frac{1}{h_1}\right),$$

it follows that K has positive entries in all positions where either I or T have positive entries. Since t^2, $(1-t)^2$ and $t(1-t)$ are nonnegative for $t \in [0, 1]$ we have that

$$\| W(t, A)\|_\infty = \max_{1 \leqslant i \leqslant n} \left[t^2 + (1-t)^2 + t(1-t)\, h_i \sum_{j=1}^{n} \left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right)\right]$$

$$= t^2 + (1-t)^2 + 2t(1-t) \sum_{j=1}^{n} \frac{1}{h_j} (\max_{1 \leqslant i \leqslant n} h_i)$$

$$\geqslant t^2 + (1-t)^2 + 2t(1-t) \cdot n$$

with equally if and only if $h_j = \max_{1 \leqslant i \leqslant n} h_i$ for $j = 1, \dots, n$,

$$\geqslant (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + 2(\tfrac{1}{2})(\tfrac{1}{2})\, n = \tfrac{1}{2}(n+1). \quad \blacksquare$$

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